Consider the system $\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0,$

Where t denotes time and the vector $\mathbf{x}(t) = (x_1(t), x_2(t), ..., x_m(t))$ depends on time. In applic x(t) is called a state vector as it describers the state of a variable quantity at different time intervals, that is the vector **x** evolves with time. The transformation is linear and the system is called Discrete Dynamical System and it says nothing about the state vector outside the positive integer values of t.

Note that if t is in minutes, for example, then

$\mathbf{x}(1) = \mathbf{A}\mathbf{x}(0)$, the state of $\mathbf{x}$ at $t = 1$ depends on the state of $\mathbf{x}$ at $t = 0$.

$$\text{and} \quad \begin{aligned} \mathbf{x}(2) &= \mathbf{A}\mathbf{x}(1) \\ &= \mathbf{A}\mathbf{A}\mathbf{x}(0) \\ &= \mathbf{A}^2\mathbf{x}(0) \end{aligned}$$

so that after t minutes $\mathbf{x}(t+1) = \mathbf{A}^t\mathbf{x}(t)$

Assume that the matrix **A** has a complete set of eigenvectors that span $\Re^n$.
ie

$$\mathbf{A}[v_1 \quad v_2 \quad ... \quad v_n] = [\lambda_1 v_1 \quad \lambda_2 v_2 \quad ... \quad \lambda_n v_n]$$

which implies that

$$\mathbf{A}^t\mathbf{P} = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & ... & v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1^t & 0 & . & 0 \\ 0 & \lambda_2^t & . & 0 \\ . & . & . & . \\ 0 & 0 & . & \lambda_n^t \end{bmatrix}$$

Or $\mathbf{A}^t\mathbf{P} = \mathbf{P}\mathbf{D}^t$

*If v_i is an eigenvector of **A** that corresponds to λ_i then then v_i is an eigenvector of $\mathbf{A}^n$ that corresponds to λ_{in}*

Express the state vector $\mathbf{x}(t)$ as a linear combinations of the eigenvect $\mathbf{A}^t$ f

$$\mathbf{x}(t) = c_1\lambda_1^t v_1 + c_2\lambda_2^t v_2 + ... + c_n\lambda_n^t v_n$$

$$\text{Hence } \mathbf{x}_0 = c_1 v_1 + c_2 v_2 + ... + c_n v_n$$

In Matrix form

$$\begin{aligned} &\mathbf{x} = \mathbf{P}\mathbf{D}^t\mathbf{C}, \text{ and } \mathbf{x}(0) = PC \\ &\mathbf{C} = \mathbf{P}^{-1}\mathbf{x}(0) \\ &\mathbf{x}(t) = \mathbf{P}\mathbf{D}^t\mathbf{P}^{-1}\mathbf{x}_0 \end{aligned}$$

Expanded in matrix form

$$\mathbf{x}(t) = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & .... & v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & . & 0 \\ 0 & \lambda_2 & . & 0 \\ . & . & . & . \\ 0 & 0 & . & \lambda_n \end{bmatrix}^t \begin{bmatrix} c_1 \\ c_2 \\ . \\ c_n \end{bmatrix}, \mathbf{x}(0) = \mathbf{PC}$$

$$\mathbf{x}(t) = \mathbf{P} \begin{bmatrix} \lambda_1 & 0 & . & 0 \\ 0 & \lambda_2 & . & 0 \\ . & . & . & . \\ 0 & 0 & . & \lambda_n \end{bmatrix}^t \mathbf{P}^{-1}\mathbf{x}_0, \text{ with } \mathbf{x}_0 = \mathbf{PC}$$

Problems:
The Fibonacci sequence 1,1,2,3,5,8,13,21,....... evolves in positions of 0,1,2,3,4,5,...

If we write $F_k + F_{k+1} = F_{k+2}$
then this equation replaced with two equations

$$F_k + F_{k+1} = F_{k+2}$$
$$F_{k+2} = F_{k+2}$$

contains the first equation and an identity, can be written a matrix form

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix}$$

The system with initial vector is

$$\mathbf{AU}_k = \mathbf{U}_{k+1}, \quad \mathbf{U}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

in terms of the initial vector, $\mathbf{U}_k = \mathbf{A}^k \mathbf{U}_0$

The eigenvalues for and eigenvectors of **A** are

$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \lambda_2 = \frac{1-\sqrt{5}}{2}, \ \mathbf{x}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

Problem 1:

a. Find the 10^{th} Fibonacci number.

b. Find $\lim_{k\to\infty} \mathbf{U}$

Problem 2

Suppose the population in California move outside of the state at a rate of 8% per year and that people from outside the state move in at a rate of 4% per year. Assuming the population inside and outside the state remain constant, what is the distribution of the population in California k years later? What is the distribution of the population as k becomes infinite?

Let the population at time t = 0 inside and outside be z_0 and y_0 respectively.

a) Write a set of equations for the population inside and outside California after one year.

b) Convert the equation in part a into a matrix equation.

c) Write a matrix equation for the distribution of the population after k years using the initial population. (y_0 , z_0)

d) Find a matrix P that diagonalizes the matrix in part c.

e) Find an expression for (y_k , z_k)

c) Discuss $\lim_{k\to\infty}(y_k, z_k)$